NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS ADVANCED

2020 Year 12 Course Assessment Task 4 (Trial HSC Examination)
Thursday, 20 August 2020

General instructions

- Working time – 3 hour.
  (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- NESA approved calculators may be used.
- Attempt **all** questions.

SECTION I

- Mark your answers on the answer grid provided (on page 28)

SECTION II

- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

NESA STUDENT #: .... **# BOOKLETS USED**: .....

Class (please ✔)

◯ 12MAX.3 – Mr Sekaran

◯ 12MAX.5 – Mr Siu

◯ 12MAX.4 – Mrs Bhamra

◯ 12MAA.6 – Mr Ho

Marker's use only.

QUESTION	1-10	11-14	15-18	19-22	23-25	26-29	30-32	Total	%
MARKS	/10	/15	/15	/15	/15	/15	/15	/100	

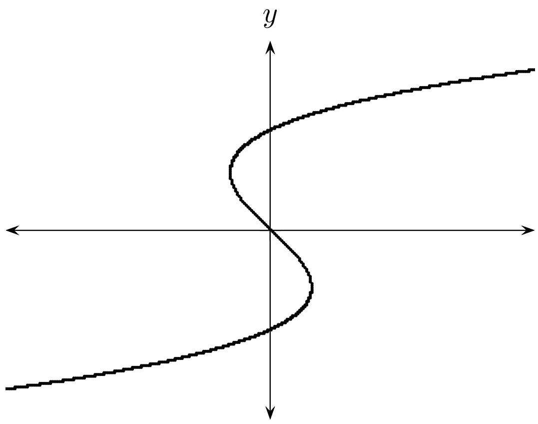
Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer grid provided (labelled as page 28).

Questions **Marks**

1. Which of the following describes the curve shown below? **1**



(A) one-to-one

(B) many-to-one

(C) one-to-many

(D) many-to-many

2. What is the value of x for which $4^{3-5x} = 8^{2x}$? **1**

(A) $\frac{1}{3}$

(B) $\frac{3}{8}$

(C) $-\frac{3}{7}$

(D) $-\frac{3}{8}$

3. What is the value of a for which the y-axis is tangent to the circle? **1**

$$(x-2)^2 + (y-5)^2 = a$$

(A) 2

(B) 4

(C) 5

(D) 25

4. Which one of the following is the set of all solutions to $2x^2 - 5x + 2 \geq 0$? **1**

(A) $\left[\frac{1}{2}, 2\right]$

(B) $\left(\frac{1}{2}, 2\right)$

(C) $\left(-\infty, \frac{1}{2}\right) \cup (2, \infty)$

(D) $\left(-\infty, \frac{1}{2}\right] \cup [2, \infty)$

5. What is the value of $\tan x$ given that $\sin x = \dfrac{1}{3}$ for $\dfrac{\pi}{2} < x < \pi$. **1**

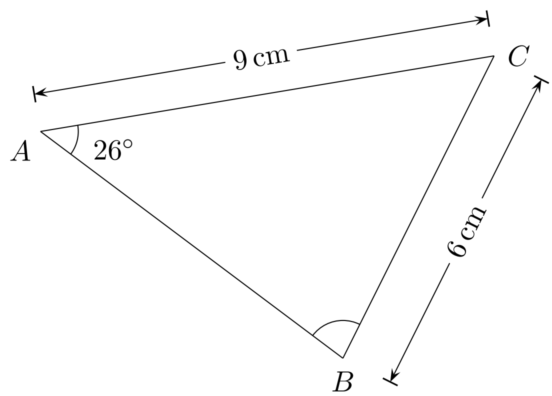
(A) $-\dfrac{1}{2\sqrt{2}}$

(B) $\dfrac{2\sqrt{2}}{3}$

(C) $-\dfrac{3}{2\sqrt{2}}$

(D) $-2\sqrt{2}$

6. What are the possible values of $\angle ABC$, correct to the nearest minute? **1**



(NOT TO SCALE)

(A) 41°

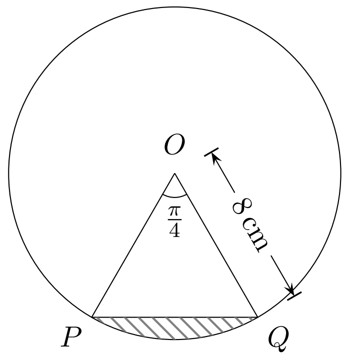
(B) 17°

(C) 17°, 163°

(D) 41°, 139°

Examination continues overleaf...

7. What is the area of the shaded segment? **1**



(NOT TO SCALE)

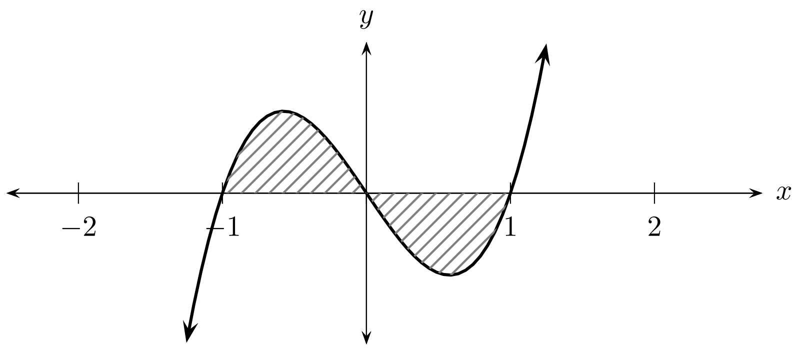
(A) $32\left(\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right)$ cm^2

(B) $32\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)$ cm^2

(C) $64\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)$ cm^2

(D) $16\left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)$ cm^2

8. The diagram shows the area bounded by the graph of $f(x) = x^3 - x$ and the x-axis. **1**



Given that $f(x)$ is an odd function, which of the following correctly gives the shaded area?

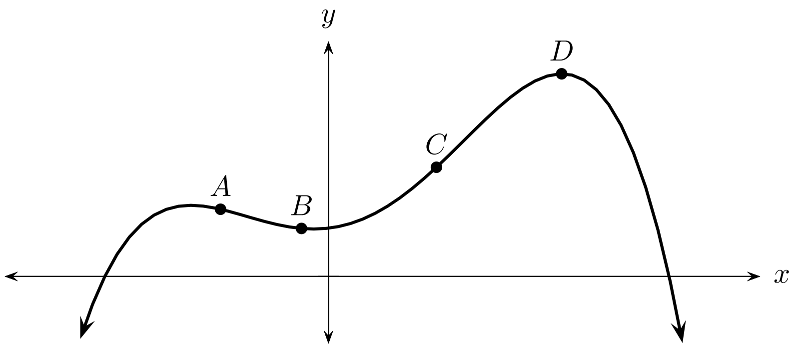
(A) $2\int_{-1}^{0} \left(x^3 - x\right)\, dx$

(B) $2\int_{0}^{1} \left(x^3 - x\right)\, dx$

(C) $\int_{-1}^{1} \left(x^3 - x\right)\, dx$

(D) $2\int_{-1}^{1} \left(x^3 - x\right)\, dx$

9. Among the labelled points on this curve, at which point is the rate of change of y greatest? **1**



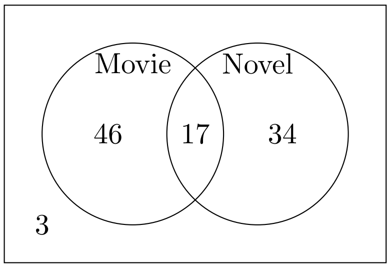
(A) A

(B) B

(C) C

(D) D

10. A magazine surveyed 100 readers about whether they had read a certain novel or watched its movie adaptation. The data collected is represented in the Venn diagram below. **1**



A reader was selected at random. If the reader selected has watched the movie, what is the probability that they had also read the novel?

(A) $\dfrac{17}{100}$

(B) $\dfrac{17}{63}$

(C) $\dfrac{17}{46}$

(D) $\dfrac{17}{51}$

Examination continues overleaf...

Section II

90 marks
Attempt Question 11 to 32
Allow approximately 2 hours and 45 minutes for this section

Write your answers in the space provided.

Question 11 (2 Marks)

Given the function $f(x) = x^2 - 4x$ and $g(x) = x^2 - 9$, write an expression for $g\Big(f(x)\Big)$ in its simplest form. **2**

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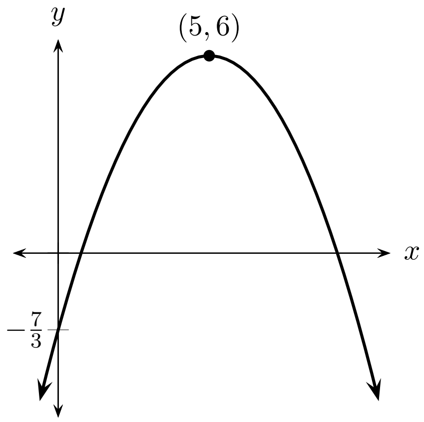
Question 12 (3 Marks)

The function $f(x) = x^2$ has been transformed into a new function whose graph is shown below: **3**



Find the equation of the new function in the form $g(x) = kf(x + b) + c$ for some constants k, b and c.

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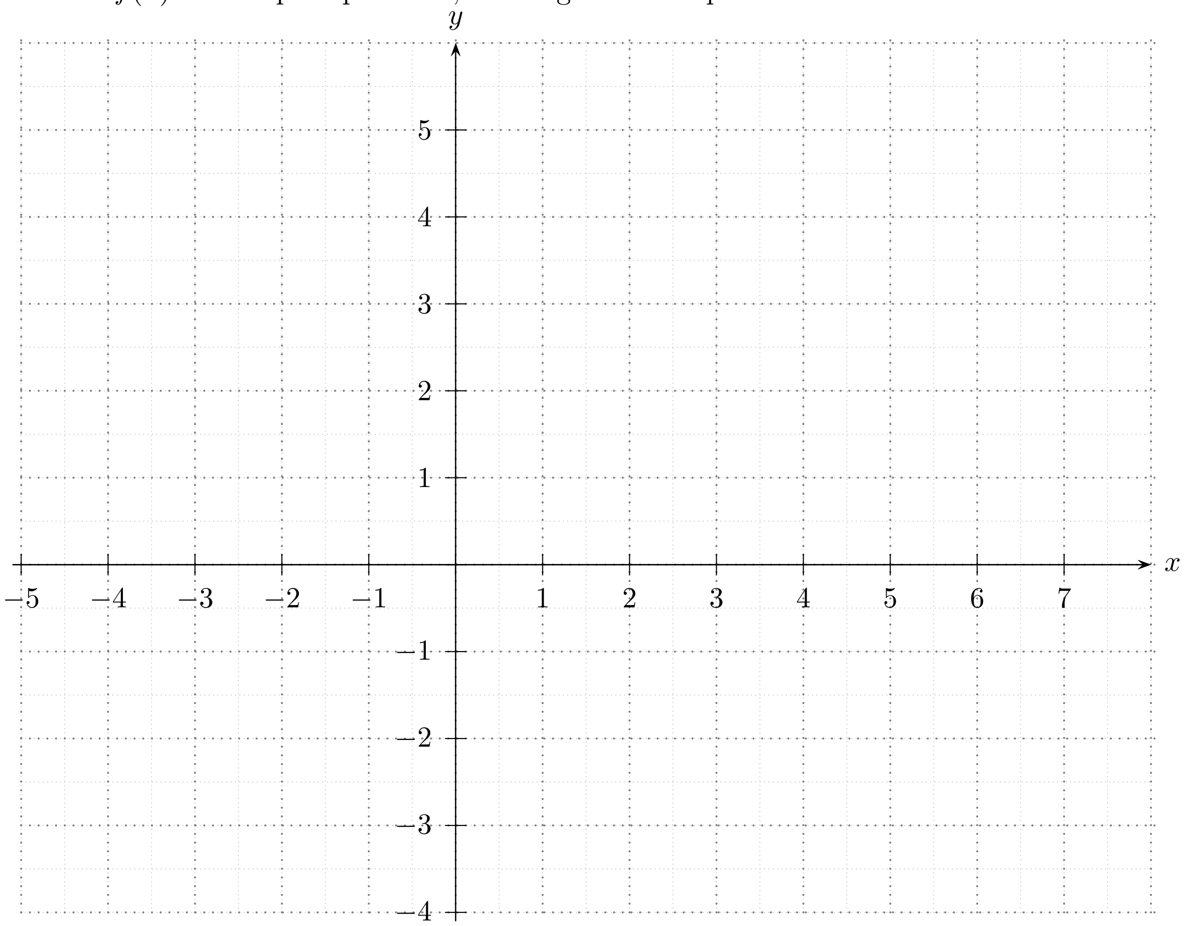
Question 13 (6 Marks)

A function is defined as $f(x) = \begin{cases} \log_2(x+4) & -4 < x \leq 4 \\ 2\,|x-5|+1 & 4 < x < 7 \\ 5 & x \geq 7 \end{cases}$

(a) Sketch $f(x)$ in the space provided, showing all intercepts with the axes. **4**



(b) Using interval notation, state the domain and range of $f(x)$. **2**

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Examination continues overleaf...

Question 14 (4 Marks)

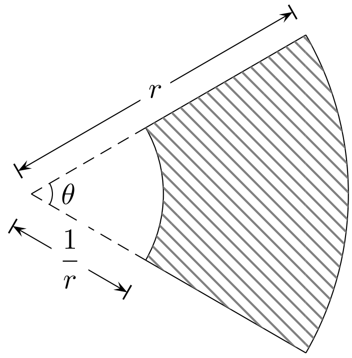
A sequence is given by $\sqrt{8}, \sqrt{32}, \sqrt{72}, \cdots$

(a) Show that the terms form an arithmetic sequence. **1**

(b) The sum of the first n terms is $132\sqrt{2}$. Find the value of n. **3**

Question 15 (3 Marks)

An annulus sector is made with an angle θ at its centre. Its inner and outer radii are $\frac{1}{r}$ metres and r metres respectively, as shown. **3**



(NOT TO SCALE)

The perimeter of this annulus sector is 6 metres.

Show that $\theta = \dfrac{2(-r^2 + 3r + 1)}{r^2 + 1}$.

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Examination continues overleaf...

Question 16 (3 Marks)

Prove that $2\cos^2\theta + 2\sec^2\left(\dfrac{\pi}{2} - \theta\right) + 2\sin^2\theta = 4 + 2\cot^2\theta$. **3**

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Question 17 (3 Marks)

Solve $4 - 8\sin^2 2\theta = 0$ for $0 \leq \theta \leq 2\pi$. **3**

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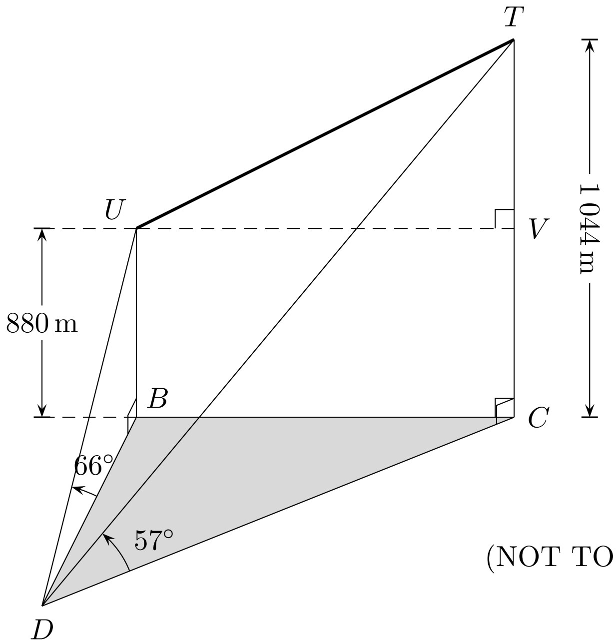
Question 18 (6 Marks)

David stands on ground level at point D and observes a cable car travel along a slanted ropeway. The cable car departs from Station U at an altitude of 880 m and arrives at Station T at an altitude of 1 044 m. David observes Station U at an angle of elevation of 66° and Station T at an angle of elevation of 57°.



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(a) Show that $DB = 880 \cot 66°$ and find a similar expression for DC. **2**

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(b) The area of $\triangle BDC$ is $112\,860\,\text{m}^2$. Show that $\angle BDC = 58°11'$. **2**

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Examination continues overleaf...

(c) Hence find the slant length of the ropeway UT, correct to two decimal places. **2**

Question 19 (3 Marks)

A discrete random variable X has the probability distribution table shown. **3**

x	1	2	3	4
$P(X = x)$	0.35	a	0.2	b

Find the value of a and b given that $E(X) = 2.15$.

Examination continues overleaf...

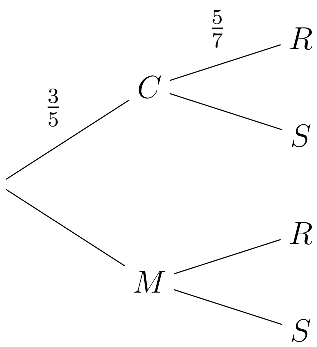
Question 20 (3 Marks)

A factory produces chocolate chip cookies and oatmeal cookies in both regular and sugar free variants. Each day, $\dfrac{3}{5}$ of the cookies produced are chocolate chip and the rest are oatmeal. According to demand, $\dfrac{5}{7}$ of the chocolate chip cookies are regular and the rest of the chocolate chip cookies are sugar free, while $\dfrac{2}{3}$ of the oatmeal cookies are sugar free and the rest of the oatmeal cookies are regular.

(a) Using the information supplied, label all remaining branches of the following probability tree with the correct probabilities. **1**

The symbols given in the tree are:

- C represents the chocolate chip cookie.
- M represents the oatmeal cookie.
- S represents a sugar free cookie.
- R represents a regular cookie.



(b) A cookie is selected at random. Given that a sugar free cookie is selected, find the probability that the cookie was chocolate-chip. **2**

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Question 21 (3 Marks)

A company tracked the number of sales made by each sales staff in a day. Using the data from a large number of days, the company estimated the probability of the number of sales made by each sales staff in a day. **3**

Let the random variable S be the number of sales made by Linda per day.
The following table shows the probability data for Linda's daily sales.

s	2	3	4	5	6	7
$P(S = s)$	0.05	0.1	0.15	0.35	0.15	0.2

Linda is expected to make 5.05 sales per day. Find Linda's variance and standard deviation, correct to two decimal places.

(Hint: Add any required rows in the table using the space provided above.)

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Examination continues overleaf...

Question 22 (6 Marks)

An ice cream company launches a promotion in which customers have a chance of winning a limited edition item. The probability of winning this prize is 0.07 with each individual purchase. Each person can only win the prize once.

Benjamin decides to purchase one ice block each day until he wins the prize.

(a) Find the probability that he wins the item on the first or second day, correct to two decimal places. **2**

(b) Hence find the smallest number of days for which the probability that Benjamin will win the limited edition item, is greater than 80%. **4**

Question 23 (4 Marks)

A function is given by $f(x) = -(x-2)(x+2)(x-1)$.

(a) Sketch the curve in space provided without using calculus. Show all intercepts with the axes. **1**

(b) Hence find the exact area bounded by the curve and the x-axis. **3**

Examination continues overleaf…

Question 24 (7 Marks)

A function is given by $f(x) = -\dfrac{1}{2}x^4 + 2x^3 + \dfrac{3}{2}$.

(a) Find the coordinates of stationary points and determine their nature. **3**

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(b) Show that there is another point of inflexion at $x = 2$. **2**

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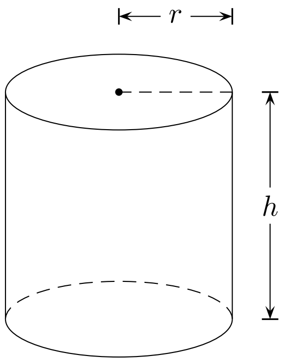
(c) Hence sketch the curve in the space provided, showing all stationary points and points of inflexion. (Do not find the x-intercepts) **2**

Examination continues overleaf...

Question 25 (4 Marks)

A company is designing a closed cylindrical container of radius r cm and height h cm for their new product. The container is to have a surface area of 112π cm^2.



(NOT TO SCALE)

(a) Show that the volume of the container is given by the expression: **2**

$$V = 56\pi r - \pi r^3$$

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(b) Find the radius for which the volume of the container is maximised, correct to two decimal places. **2**

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Question 26 (2 Marks)

Find the equation of the tangent to $y = e^{-\sin x}$ at $x = \pi$. **2**

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Question 27 (3 Marks)

The population P of a colony of ants is given by $P(t)$, where t is the time in years after a scientist first discovered the colony. The ant population changes at a rate modelled by the function: **3**

$$\frac{dP}{dt} = 1800e^{0.6t}$$

The colony had a population of 700 ants when first discovered by the scientist. Calculate the number of years for the population to reach 79 700 ants, correct to two decimal places.

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Examination continues overleaf...

Question 28 (4 Marks)

(a) Differentiate $y = 2\left(7 - e^{x^3}\right)^5$ **2**

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(b) Hence find $\int 5x^2 e^{x^3}\left(7 - e^{x^3}\right)^4\, dx$. **2**

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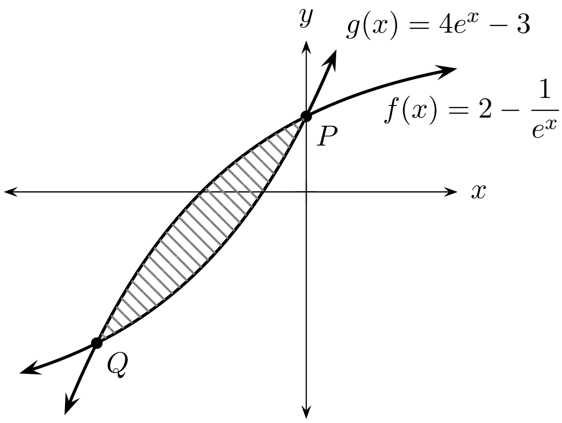
Question 29 (6 Marks)

The functions $f(x) = 2 - \dfrac{1}{e^x}$ and $g(x) = 4e^x - 3$ intersect at points P and Q as shown.



(a) Show that the curves intersect when $4e^{2x} - 5e^x + 1 = 0$. **1**

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(b) Show that the points of intersection occur at $x = 0$ and $x = \ln \dfrac{1}{4}$. **2**

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(c) Hence find the area of the region bounded by $f(x)$ and $g(x)$, correct to three decimal places. **3**

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Examination continues overleaf...

Question 30 (4 Marks)

(a) Show that $\dfrac{d}{dx}(\sec x) = \dfrac{\sin x}{\cos^2 x}$. **1**

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(b) Hence show that the derivative of $y = \dfrac{\sec x}{\sin x + \cos x}$ can be expressed as: **3**

$$\frac{dy}{dx} = \frac{\tan^2 x + 2\tan x - 1}{1 + 2\sin x \cos x}$$

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Question 31 (3 Marks)

Find the primitive function $F(x)$ of the function $f(x) = \cot x$, given that $F\left(\frac{\pi}{2}\right) = 3$. **3**

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Examination continues overleaf...

Question 32 (8 Marks)

The velocity of a particle is given by:

$$v(t) = -6\cos 2t \qquad \text{for } 0 \leq t \leq 2\pi$$

where v is measured in metres per second and time is measured in seconds.

After $\dfrac{\pi}{2}$ seconds, the particle has a displacement of 4 metres.

(a) Show that the displacement of the particle is given by: **2**

$$x(t) = -3\sin 2t + 4$$

(b) Sketch the graph of $x(t)$ as a function of t for $0 \leq t \leq 2\pi$. **2**

(c) Find the acceleration of the particle in terms of t, and hence state the maximum magnitude of acceleration. **2**

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(d) State the values of t for which the displacement of the particle is increasing at an increasing rate. **2**

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End of paper.

Suggested Band 6 Responses

1. (C) **2.** (B) **3.** (B) **4.** (D) **5.** (A)
6. (D) **7.** (B) **8.** (A) **9.** (C) **10.** (B)

Question 11 (2 marks)

✓ [1] for correctly forming $g(f(x))$.
✓ [1] for final answer.

$$\begin{aligned} g\left(f(x)\right) &= g(x^2-4x) \\ &= (x^2-4x)^2-9 \\ \therefore g\left(f(x)\right) &= x^4-8x^3+16x^2-9 \end{aligned}$$

Question 12 (3 marks)

✓ [1] for correct values of b and c.
✓ [1] for finding k using y-intercept.
✓ [1] for final equation of $g(x)$.

$$g(x) = k(x+b)^2+c$$

Substitute vertex (5, 6):

$$y = k(x-5)^2+6$$

Substitute $(0, -\frac{7}{3})$:

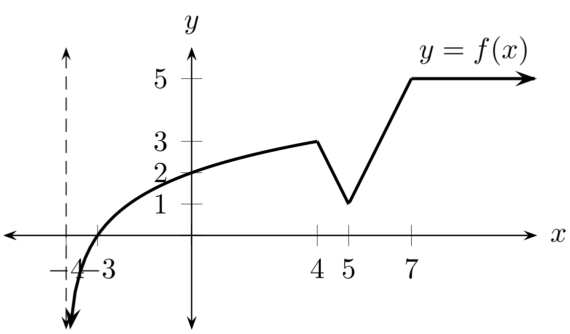
$$\begin{aligned} -\frac{7}{3} &= k(0-5)^2+6 \\ k &= -\frac{1}{3} \end{aligned}$$

$$\therefore f(x) = -\frac{1}{3}f(x-5)+6$$

Question 13

(a) (4 marks)

✓ [1] for correct axes intercepts.
✓ [1] for logarithm shape with asymptote.
✓ [1] for absolute value shape with vertex.
✓ [1] for horizontal line.



(b) (2 marks)

✓ [1] for correct domain.
✓ [1] for correct range.

$$\begin{aligned} D &: \{x : x \in (-4, \infty)\} \\ R &: \{y : y \in (-\infty, 5]\} \end{aligned}$$

Question 14

(a) (1 mark)

✓ [1] for correct proof.

Sequence can be expressed as:
$2\sqrt{2}$, $4\sqrt{2}$, $6\sqrt{2}$, $\cdots$

$$\begin{aligned} T_3-T_2 &= 6\sqrt{2}-4\sqrt{2} \\ &= 2\sqrt{2} \\ T_2-T_1 &= 4\sqrt{2}-2\sqrt{2} \\ &= 2\sqrt{2} \\ T_3-T_2 &= T_2-T_1 \end{aligned}$$

Hence arithmetic sequence.

(b) (3 marks)

✓ [1] for forming equation for arithmetic sequence sum.
✓ [1] for correct manipulation to quadratic formula.
✓ [1] for final answer and disregarding negative solution.

$$\begin{aligned} S_n &= 132\sqrt{2} \\ 132\sqrt{2} &= \frac{n}{2}\left(2\times 2\sqrt{2}+(n-1)\times 2\sqrt{2}\right) \\ 132\sqrt{2} &= n\left(2\sqrt{2}+n\sqrt{2}-\sqrt{2}\right) \\ 132\sqrt{2} &= n^2\sqrt{2}+n\sqrt{2} \end{aligned}$$

$$\begin{aligned} n^2+n-132 &= 0 \\ (n-11)(n+12) &= 0 \end{aligned}$$

$$n = 11 \text{ or } n = -12$$

But $n \geq 1$, hence $n = 11$ only.

Question 15 (3 marks)

- ✓ [1] for forming equation for perimeter.
- ✓ [1] for significant progress in simplifying equation.
- ✓ [1] for final result.

$$P = 2\left(r - \frac{1}{r}\right) + \frac{1}{r}\theta + r\theta$$

$$\frac{2(r^2-1)}{r} + \frac{\theta}{r} + \frac{r^2\theta}{r} = 6$$

$$2r^2 - 2 + \theta + r^2\theta = 6r$$

$$\begin{aligned}\theta(1+r^2) &= -2r^2 + 6r + 2\\ \therefore \theta &= \frac{2(-r^2+3r+1)}{r^2+1}\end{aligned}$$

Question 16 (3 marks)

- ✓ [1] for using trigonometric identity.
- ✓ [1] for using complementary angle relationship.
- ✓ [1] for final result.

$$\begin{aligned}\text{LHS} &= 2(\sin^2 x + \cos^2 x) + 2\,\text{cosec}^2\, x\\ &= 2 + 2(1 + \cot^2 x)\\ &= 4 + 2\cot^2 x\\ \therefore \text{LHS} &= \text{RHS}\end{aligned}$$

Question 17 (3 marks)

- ✓ [1] for correct manipulation to $\sin 2\theta$.
- ✓ [1] for solutions within domain of $0 \le 2\theta \le 4\pi$.
- ✓ [1] for final answers.

$$\begin{aligned}\sin^2 2\theta &= \frac{1}{2} \qquad \text{for } 0^\circ \le 2\theta \le 4\pi\\ \sin 2\theta &= \pm\frac{1}{\sqrt{2}}\\ \text{ref } \angle &= \frac{\pi}{4}\\ 2\theta &= \frac{\pi}{4}, \pi - \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4},\\ &\qquad 2\pi + \frac{\pi}{4}, 3\pi - \frac{\pi}{4}, 3\pi - \frac{\pi}{4}, 4\pi - \frac{\pi}{4}\\ &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}\\ \therefore \theta &= \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}\end{aligned}$$

Question 18

(a) (2 marks)

- ✓ [1] for correct expression for DB.
- ✓ [1] for correct expression for DC.

In $\triangle DBU$:
$$\begin{aligned}\tan 66^\circ &= \frac{880}{DB}\\ DB &= \frac{880}{\tan 66^\circ}\\ \therefore DB &= 880\cot 66^\circ\end{aligned}$$

In $\triangle DTC$:
$$\begin{aligned}\tan 57^\circ &= \frac{1044}{DC}\\ DC &= \frac{1044}{\tan 57^\circ}\\ \therefore DC &= 1044\cot 57^\circ\end{aligned}$$

(b) (2 marks)

- ✓ [1] for forming equation for area.
- ✓ [1] for final answer.

In $\triangle DBC$:

$$A = \frac{1}{2}(880\cot 66^\circ)(1044\cot 57^\circ)\sin\angle BDC$$
$$112860 = 459360\cot 66^\circ \cot 57^\circ \sin\angle BDC$$

$$\begin{aligned}\sin\angle BDC &= \frac{57\tan 66^\circ \tan 57^\circ}{232}\\ \therefore \angle BDC &= 58^\circ 11'\end{aligned}$$

(c) (2 marks)

✓ [1] for forming expression for BC using cosine rule.

✓ [1] for final answer.

In $\triangle DTC$:

$$\begin{aligned} BC^2 &= (880\cot 66°)^2 + (1044\cot 57°)^2 \\ &\quad - 2(880\cot 66°)(1044\cot 57°)\cos 58°11' \\ BC &= 577.13 \\ UV &= 577.13 \\ TV &= 1044 - 880 \\ &= 164 \end{aligned}$$

In $\triangle UVT$: $\quad UT^2 = (577.13)^2 + 164^2$

$$\therefore UT = 599.98$$

Hence the ropeway is 599.98 m in length.

Question 19 (3 marks)

✓ [1] for forming equation for $\Sigma p(x)$.

✓ [1] for forming equation for $E(X)$.

✓ [1] for final answers.

x	1	2	3	4
$P(X=x)$	0.35	a	0.2	b
$x \cdot p(x)$	0.35	$2a$	0.6	$4b$

$$\Sigma p(x) = 1$$
$$0.35 + a + 0.2 + b = 1$$
$$a + b = 0.45 \qquad (19.1)$$

$$E(X) = 0.35 + 2a + 0.6 + 4b$$
$$2.15 = 2a + 4b = 0.95$$

$$a + 2b = 0.6 \qquad (19.2)$$

Subtract (19.2) by (19.1):

$$b = 0.15$$

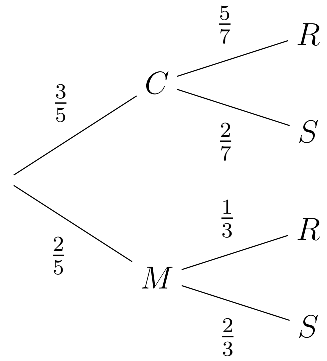
Substitute b into (19.1):

$$a + 0.15 = 0.45$$
$$\therefore a = 0.3\ ,\ b = 0.15$$

Question 20

(a) (1 mark)

✓ [1] for correctly labelled tree diagram.



(b) (2 marks)

✓ [1] for forming equation for conditional probability.

✓ [1] for final answer.

$$\begin{aligned} P(C|S) &= \frac{P(C \cap S)}{P(S)} \\ &= \frac{\frac{3}{5} \times \frac{2}{7}}{\frac{3}{5} \times \frac{2}{7} + \frac{2}{5} \times \frac{2}{3}} \\ &= \frac{6}{35} \div \frac{46}{105} \\ &= \frac{9}{23} \end{aligned}$$

Question 21 (3 marks)

✓ [1] for correct values of $s^2 p(s)$.

✓ [1] for correct value for variance.

✓ [1] for correct value for standard deviation.

s	2	3	4	5	6	7
$P(S=s)$	0.05	0.1	0.15	0.35	0.15	0.2
s^2	4	9	16	25	36	49
$s^2p(s)$	0.2	0.9	2.4	8.75	5.4	9.8

$$\begin{aligned} Var(S) &= \Sigma s^2 p(s) - E(S)^2 \\ &= (0.2 + 0.9 + 2.4 + 8.75 + 5.4 + 9.8) \\ &\quad - 5.05^2 \end{aligned}$$
$$\therefore Var(S) = 1.95$$
$$\sigma = \sqrt{1.948}$$
$$\therefore \sigma = 1.40$$

Question 22

(a) (2 marks)

- ✓ [1] for forming equation for probability.
- ✓ [1] for final answer.

Let A be the event that Benjamin wins on the first or second day.
Let W be the event that Benjamin wins.

$$\begin{aligned} P(\overline{W}) &= 0.93 \\ P(A) &= P(W) + P(\overline{W}W) \\ &= 0.07 + 0.93 \times 0.07 \\ &= 0.1351 \end{aligned}$$

Hence the probability that Benjamin wins on the first or second day is 0.14.

(b) (4 marks)

- ✓ [1] for forming equation for geometric sum.
- ✓ [2] for significant progress in solving for n.
- ✓ [1] for final answer.

Let B be the event that Benjamin wins eventually.
The sequence of probabilities that Benjamin wins on a certain day:

$$0.07,\ 0.93 \times 0.07,\ 0.93^2 \times 0.07,\ \cdots$$

Forms a geometric sequence:

$$a = 0.07\ ,\ r = 0.93$$

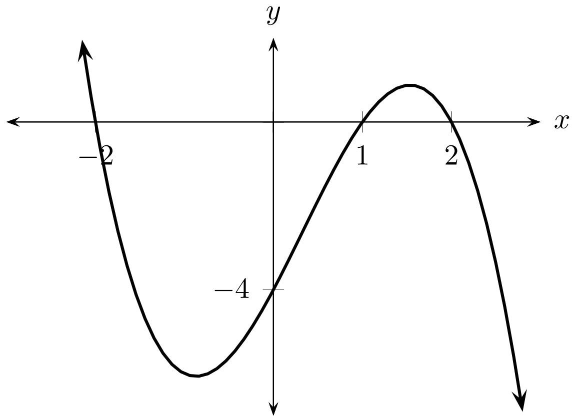
$$\begin{aligned} P(B) &> 0.8 \\ \frac{0.07(1-0.93^n)}{1-0.93} &> 0.8 \\ 1 - 0.93^n &> 0.8 \\ -0.93^n &> -0.2 \\ 0.93^n &< 0.2 \\ n &> \log_0 .930.2 \\ n &> \frac{\log_1 00.2}{\log_1 00.93} \\ n &> 22.18 \end{aligned}$$

Hence the smallest number of days is 23.

Question 23

(a) (1 mark)

- ✓ [1] for final graph with correct axes intercepts.



$f(x) = -(x-2)(x+2)(x-1)$

(b) (3 marks)

- ✓ [1] for correct integral of $f(x)$.
- ✓ [1] for forming equation for area using signed areas.
- ✓ [1] for final answer.

$$\begin{aligned} f(x) &= -(x^2-4)(x-1) \\ \therefore f(x) &= -(x^3 - x^2 - 4x + 4) \end{aligned}$$

Considering the negative signed area for $-2 \le x \le 1$:

$$\begin{aligned} \text{Area} &= -\int_{-2}^{1} \left[-(x^3 - x^2 - 4x + 4)\right]\, dx \\ &\quad + \int_{1}^{2} \left[(-(x^3 - x^2 - 4x + 4)\right]\, dx \\ &= \int_{-2}^{1} \left(x^3 - x^2 - 4x + 4\right)\, dx \\ &\quad - \int_{1}^{2} \left(x^3 - x^2 - 4x + 4\right)\, dx \\ &= \left[\frac{x^4}{4} - \frac{x^3}{3} - 2x^2 + 4x\right]_{-2}^{1} \\ &\quad - \left[\frac{x^4}{4} - \frac{x^3}{3} - 2x^2 + 4x\right]_{1}^{2} \\ &= [F(1) - F(-2)] - [F(2) - F(1)] \\ &= \left(\frac{23}{12} - \frac{-28}{3}\right) - \left(\frac{4}{3} - \frac{23}{12}\right) \end{aligned}$$

$$\therefore \text{Area} = \frac{71}{6}$$

Question 24

(a) (3 marks)

- ✓ [1] for correct coordinates for both stationary points.
- ✓ [1] for correctly testing nature of both stationary points.
- ✓ [1] for correctly stating the nature of both stationary points.

$$f'(x) = -2x^3 + 6x^2$$
$$f''(x) = -6x^2 + 12x$$

Stationary points occur when $f'(x) = 0$:

$$-2x^3 + 6x^2 = 0$$
$$-2x^2(x-3) = 0$$
$$x = 0 \text{ or } x = 3$$

- When $x = 0$:

$$f(0) = \frac{3}{2}$$
$$f''(0) = 0$$

Potential point of inflexion at $x = 0$.

x	-1	0	1
$f'(x)$	8	0	4
m	/	-	/

Hence $(0, 1\frac{1}{2})$ is a horizontal point of inflexion.

- When $x = 3$:

$$f(3) = 15$$
$$f''(3) = -18$$
$$f''(3) < 0$$

Hence $(3, 15)$ is a maximum stationary point.

(b) (2 marks)

- ✓ [1] for final graph with correct axes intercepts.

Points of inflexion occur when $f''(x) = 0$:

$$-6x^2 + 12x = 0$$
$$-6x(x-2) = 0$$
$$x = 0 \text{ or } x = 2$$

- Horizontal point of inflexion at $(0, 1\frac{1}{2})$ from part (a).

- When $x = 2$:

$$f(2) = \frac{19}{2}$$

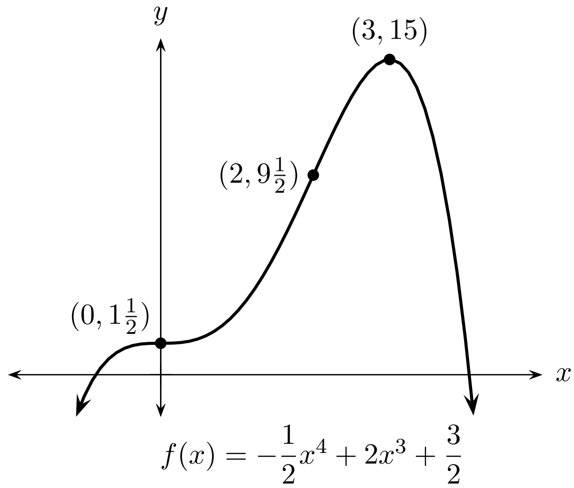
x	1	2	3
$f''(x)$	6	0	-18

$\therefore$ Change in concavity at $x = 0$.

Hence $(2, 9\frac{1}{2})$ is a point of inflexion.

(c) (2 marks)

- ✓ [1] for correct shape of graph.
- ✓ [1] for correctly shown stationary points and points of inflexion.



Question 25

(a) (2 marks)

✓ [1] for correct manipulation to expression for h.

✓ [1] for final result.

Let SA be the surface area of the cylinder.

$$\begin{aligned} SA &= 2\pi r^2 + 2\pi rh \\ 112\pi &= 2\pi r^2 + 2\pi rh \\ 112\pi - 2\pi r^2 &= 2\pi rh \\ h &= \frac{56 - r^2}{r} \end{aligned}$$

$$\begin{aligned} V &= \pi r^2 \left(\frac{56 - r^2}{r}\right) \\ \therefore V &= 56\pi r - \pi r^3 \end{aligned}$$

(b) (2 marks)

✓ [1] for correct value for r.

✓ [1] for testing that volume is maximised.

$$\begin{aligned} \frac{dV}{dr} &= 56\pi - 3\pi r^2 \\ \frac{d^2V}{dr^2} &= -6\pi r \end{aligned}$$

Turning points for V occur when $\frac{dV}{dr} = 0$:

$$56\pi - 3\pi r^2 = 0$$

$$\begin{aligned} r^2 &= \frac{56}{3} \\ r &= \sqrt{\frac{56}{3}}\text{, since } r \geq 0 \end{aligned}$$

When $r = \sqrt{\frac{56}{3}}$:

$$\begin{aligned} \frac{d^2V}{dr^2} &= -6\pi\sqrt{\frac{56}{3}} \\ \therefore \frac{d^2V}{dr^2} &< 0 \end{aligned}$$

Hence the volume is maximised when $r = 4.32$ cm.

Question 26 (2 marks)

✓ [1] for correct gradient at $x = \pi$.

✓ [1] for final answer.

$$\frac{dy}{dx} = -\cos x \cdot e^{-\sin x}$$

When $x = \pi$:

$$\begin{aligned} y &= 1 \\ \frac{dy}{dx} &= 1 \end{aligned}$$

$\therefore m = 1$ at $(\pi, 1)$.

Using the point-gradient form of a straight line:

$$\begin{aligned} y - 1 &= 1 \times (x - \pi) \\ \therefore y &= x - \pi + 1 \end{aligned}$$

Hence the equation of the tangent is $y = x - \pi + 1$

Question 27 (3 marks)

✓ [1] for correct integral of $\frac{dP}{dt}$.

✓ [1] for significant progress in solving for t.

✓ [1] for final answer.

$$\begin{aligned} P &= \int 1800e^{0.6t}\,dx \\ &= 3000e^{0.6t} + c \end{aligned}$$

When $t = 0,\ P = 700$:

$$\begin{aligned} 700 &= 3000e^0 + c \\ c &= -2300 \end{aligned}$$

$$\therefore P = 3000e^{0.6t} - 2300$$

When $P = 79700$:

$$\begin{aligned} 79700 &= 3000e^{0.6t} - 2300 \\ \frac{82}{3} &= e^{0.6t} \\ \ln\frac{82}{3} &= 0.6t \\ \therefore t &= \frac{5}{3}\ln\frac{82}{3} \end{aligned}$$

Hence the population will reach 79 700 after $t = 5.51$ years.

Question 28

(a) (2 marks)

✓ [1] for correct use of chain rule.

✓ [1] for final answer.

$$\frac{dy}{dx} = 10\left(7 - e^{x^3}\right)^4 \times \left(-3x^2 e^{x^3}\right)$$
$$\therefore \frac{dy}{dx} = -30x^2 e^{x^3}\left(7 - e^{x^3}\right)^4$$

(b) (2 marks)

✓ [1] for correct use of part (a).

✓ [1] for final answer.

From part (a):

$$\int -30x^2 e^{x^3}\left(7 - e^{x^3}\right)^4\,dx = \int \frac{dy}{dx}\,dx$$
$$\int -30x^2 e^{x^3}\left(7 - e^{x^3}\right)^4\,dx = 2\left(7 - e^{x^3}\right)^5 + c$$

Divide both sides by -6:

$$\therefore \int 5x^2 e^{x^3}\left(7 - e^{x^3}\right)^4\,dx = -\frac{1}{3}\left(7 - e^{x^3}\right)^5 + c$$

(c) (3 marks)

✓ [1] for forming integral for area between $f(x)$ and $g(x)$.

✓ [1] for significant progress in finding area.

✓ [1] for final answer.

$$A = \int_{\ln\frac{1}{4}}^{0} \left[\left(2 - \frac{1}{e^x}\right) - (4e^x - 3)\right]\,dx$$
$$= \int_{\ln\frac{1}{4}}^{0} \left(5 - e^{-x} - 4e^x\right)\,dx$$
$$= \left[5x + e^{-x} - 4e^x\right]_{\ln\frac{1}{4}}^{0}$$
$$= \left(0 + e^0 - 4e^0\right) - \left(5\ln\frac{1}{4} + e^{-\ln\frac{1}{4}} - 4e^{\ln\frac{1}{4}}\right)$$
$$= (-3) - \left(5\ln\frac{1}{4} + 4 - 1\right)$$
$$= -5\ln\frac{1}{4} - 6$$
$$\therefore A = 5\ln 4 - 6$$

Hence the shaded area is 0.931 units^2.

Question 29

(a) (1 mark)

✓ [1] for correct process.

Equate $f(x)$ and $g(x)$:

$$4e^x - 3 = 2 - \frac{1}{e^x}$$
$$4e^{2x} - 3e^x = 2e^x - 1$$
$$4e^{2x} - 5e^x + 1 = 0$$

(b) (2 marks)

✓ [1] for correct factorisation.

✓ [1] for final result.

$$(4e^x - 1)(e^x - 1) = 0$$

$$e^x = \frac{1}{4} \text{ or } e^x = 1$$
$$x = \ln\frac{1}{4} \text{ or } x = 0$$

Question 30

(a) (1 mark)

✓ [1] for correct manipulation to final answer.

$$\begin{aligned}\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left((\cos x)^{-1}\right)\\ &= -\left((\cos x)^{-2}\right) \times (-\sin x)\\ \therefore \frac{d}{dx}(\sec x) &= \frac{\sin x}{\cos^2 x}\end{aligned}$$

(b) (3 marks)

✓ [1] for forming derivative using quotient rule.

✓ [1] for significant progress in manipulating derivative.

✓ [1] for final result.

Let $\dfrac{u}{v} = \dfrac{\sec x}{\sin x + \cos x}$

Using the quotient rule and part (a):

$$u = \sec x \qquad v = \sin x + \cos x$$

$$u' = \frac{\sin x}{\cos^2 x} \qquad v' = \cos x - \sin x$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\left(\sin x + \cos x\right) \cdot \frac{\sin x}{\cos^2 x} - \sec x\left(\cos x - \sin x\right)}{\left(\sin x + \cos x\right)^2}\\ &= \frac{\tan^2 x + \tan x - 1 + \tan x}{\sin^2 x + \cos^2 x + 2\sin x \cos x}\\ \therefore \frac{dy}{dx} &= \frac{\tan^2 x + 2\tan x - 1}{1 + 2\sin x \cos x}\end{aligned}$$

Question 31 (3 marks)

✓ [1] for correct integral of $f(x)$ with absolute value signs.

✓ [1] for correct value for c.

✓ [1] for final answer.

$$\begin{aligned}f(x) &= \frac{\cos x}{\sin x}\\ F(x) &= \ln|\sin x| + c\end{aligned}$$

Using $F\left(\dfrac{\pi}{2}\right) = 3$:

$$\begin{aligned}F\left(\frac{\pi}{2}\right) &= \ln 1 + c\\ 3 &= c\\ \therefore F(x) &= \ln|\sin x| + 3\end{aligned}$$

Question 32

(a) (2 marks)

✓ [1] for correct integral of $v(t)$.

✓ [1] for correct manipulation to final answer.

$$\begin{aligned}x(t) &= \int (-6\cos 2t)\,dx\\ &= -6 \times \frac{1}{2}\sin 2t + c\\ \therefore x(t) &= -3\sin 2t + c\end{aligned}$$

When $t = \dfrac{\pi}{2}$, $x = 4$:

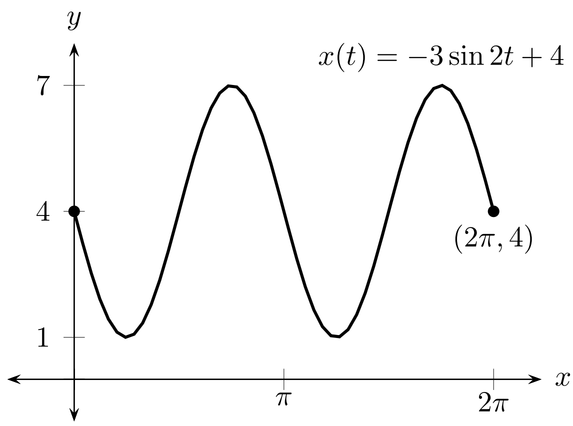
$$\begin{aligned}4 &= -3\sin\pi + c\\ \therefore c &= 4\end{aligned}$$

Hence the displacement is given by $x(t) = -3\sin 2t + 4$.

(b) (2 marks)

✓ [1] for correctly sketched shape and period.

✓ [1] for correctly sketched amplitude and endpoints.



(c) (2 marks)

✓ [1] for correct equation for acceleration.

✓ [1] for correct maximum magnitude of acceleration.

$$\begin{aligned} a(t) &= \frac{d}{dt}(-6\cos 2t) \\ &= -6(-2\sin 2t) \\ \therefore a(t) &= 12\sin 2t \end{aligned}$$

Amplitude of the acceleration graph is 12.

Hence the maximum magnitude of acceleration is 12 m s^{-2}.

(d) (2 marks)

✓ [1] for correctly stating one range of x-values.

✓ [1] for correctly stating both ranges of x-values.

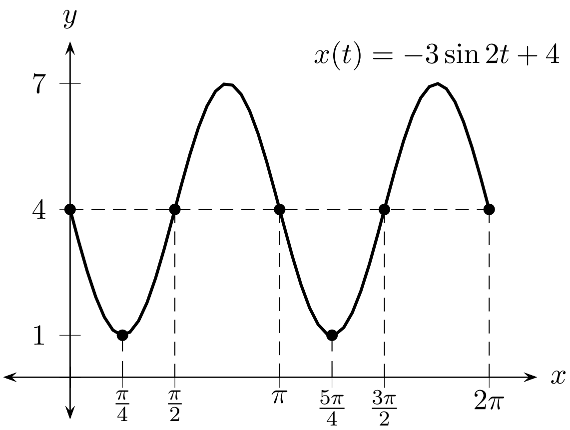
From the properties of a sine curve, the displacement graph has:

- points of inflexion at:

$$x = 0\ ,\ \frac{\pi}{2}\ ,\ \pi\ ,\ \frac{3\pi}{2}\ ,\ 2\pi$$

- minimum turning points at:

$$x = \frac{\pi}{4}\ ,\ \frac{5\pi}{4}$$



Hence the displacement of the particle is increasing at an increasing rate when $\frac{\pi}{4} < x < \frac{\pi}{2}$ and $\frac{5\pi}{4} < x < \frac{3\pi}{2}$.